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LMI Based Neurocontroller for Output–Feedback Guaranteed Cost Control of Discrete–Time Uncertain System

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Abstract—**This paper presents a nonlinear output feedback controller design method that integrates the guaranteed cost control approach for a class of discrete–time system with parametric uncertainties and neural networks (NNs). Based on the Linear Matrix Inequality (LMI) design approach, a class of output feedback controller is established, and some sufficient conditions for the existence of guaranteed cost controller is derived. The novel contribution is that the neurocontroller is substituted for the additive gain perturbations. Although the neurocontroller is included in the uncertain system, the closed–loop system is asymptotically stable and the closed–loop cost function value is not more than specified upper bound for all admissible uncertainty. A numerical example is given to illustrate the computational efficiency of the proposed method.**

I. Introduction

The output feedback problem for uncertain dynamic systems has been received much attention. For these problems, the guaranteed cost control for the uncertain discrete–time system by means of the output feedback control based on Riccati equation has been discussed in [1]. Recently, the guaranteed cost control problem for a class of the uncertain system with delay which is based on the Linear Matrix Inequality (LMI) design approach was solved by using the output feedback [2]. However, due to the presence of the design parameter for the LMI, it is known that the cost performance becomes quite large.

Neural networks (NNs) have been utilized for an intelligent control system because NNs have nonlinear mapping approximation property. Both state and output feedback neural regulators for nonlinear plants were designed [3]. As another important studies, the linear quadratic regulator (LQR) problem using multiple NNs has been investigated [4, 5]. However, these approaches may cause instability of the system, since in these researches the stability of the closed–loop system which includes the neurocontroller has not been considered.

In this paper, the output feedback guaranteed cost control problem of the discrete–time uncertain system with the neurocontroller is discussed. A static output feedback control is designed such that the cost of the system is guaranteed to be within a certain bounded for all admissible uncertainties. Firstly based on the LMI, a class of fixed static output feedback controller of the discrete–time uncertain system with the gain perturbations is newly derived. Secondly, in order to reduce the large cost performance NNs are used. The new idea is that the neurocontroller is substituted for the additive gain perturbations. As a result, although the neurocontroller is included in the discrete–time uncertain system, the robust stability of the closed–loop system and the adequate cost performance are attained. Finally, in order to demonstrate the efficiency of our design approach, the numerical example is given.

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II. An LMI–based Design Approach

Consider the following class of uncertain discrete–time linear system:

$$\begin{aligned} x(k+1) &= [A + D_1F(k)E_1]\, x(k) + Bu(k), && \text{(1a)} \\ y(k) &= Cx(k), && \text{(1b)} \\ u(k) &= [K + D_2N(k)E_2]\, y(k), && \text{(1c)} \end{aligned}$$

where $x(k) \in \Re^n$ is the state, $y(k) \in \Re^m$ is the output, $u(k) \in \Re^l$ is the control input, A, B, C, D_1, E_1, D_2 and E_2 are known constant matrices, K is the fixed gain matrix for the controller (1c), and $F(k) \in \Re^{p\times p}$ is unknown matrix function and $N(k) \in \Re^{q\times q}$ is the output of NN. It is assumed that $F(k)$ and $N(k)$ are satisfying

$$F^T(k)F(k) \le I_p,\, N^T(k)N(k) \le I_q. \quad (2)$$

Block diagram of a new proposed method is shown in Fig. 1, where L is a time lag diagram. It should be noted that the controller (1c) has the neurocontroller as the additive perturbations as $D_2N(k)E_2$.

Associated with the system (1) is the following quadratic cost function

$$J = \sum_{k=0}^{\infty} \left[x^T(k)Qx(k) + u^T(k)Ru(k)\right], \quad (3)$$

where Q and R are given as the positive definite symmetric matrices. In this situation, the definition of the guaranteed cost control with the additive gain perturbations is given below.

Definition 1 : *For the uncertain system (1) and cost function (3), if there exist a control gain matrix K and a positive scalar J^* such that for the admissible uncertainties and gain perturbations (2), the closed–loop system is asymptotically stable and the closed–loop value of the cost function (3) satisfies $J < J^*$, then J^* is said to be a guaranteed cost and K is said to be a guaranteed cost control gain matrix of the uncertain system (1) and cost function (3).*

In the above definition, it can be observed that the notation of guaranteed cost control is extended to the notation of quadratic stabilization. Moreover, the above definition is very popular for dealing with time–varying uncertainties and is also used in [6].

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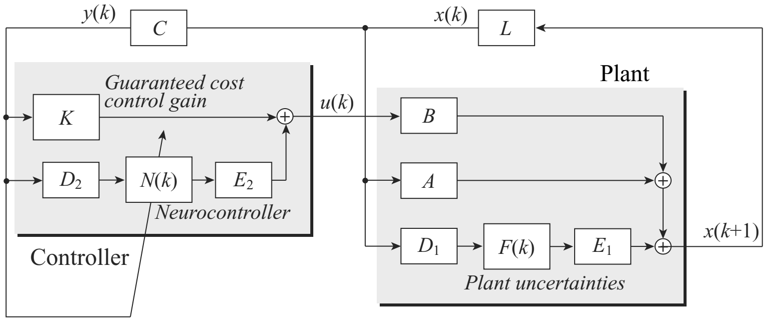

Fig. 1. Block diagram of proposed system.

The following lemma shows that the guaranteed cost control for the uncertain system (1) will define the upper bound on the cost function (3).

Lemma 1 : *Consider the following matrix inequality under the uncertain discrete–time system (1) with the cost function (3);*

$$\begin{aligned} &x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &+x^T(k)\big[Q + C^T(K + D_2N(k)E_2)^T R \\ &\times(K + D_2N(k)E_2)C\big]x(k) < 0, \end{aligned} \tag{4}$$

for all nonzero $x(k) \in \Re^n$, the uncertain matrices $F(k)$, and the gain perturbation $N(k)$.

If such condition is met, the matrix K of the controller (1c) is the guaranteed cost control gain matrix associated with the cost function (3). That is, the closed–loop uncertain system

$$\begin{aligned} x(k+1) \;=\; &[(A + D_1F(k)E_1) \\ &+B(K + D_2N(k)E_2)C]x(k) \end{aligned} \tag{5}$$

is stable and the closed–loop value of the cost function (3) satisfies

$$J < J^* = x^T(0)Px(0). \tag{6}$$

Proof: *Let us define the following Lyapunov function candidate*

$$V(x(k)) = x^T(k)Px(k). \tag{7}$$

where P is the positive definite matrix. Then, the proof can be done by using the similar technique in [7]. In this paper, it is omitted. ■

The objective of this section is to design a static output feedback gain matrix K for the discrete–time system (1) with the uncertainties and the gain perturbations (2).

Theorem 1 : *Consider the uncertain discrete–time system (1) with the cost function (3). For the uncertain matrices $F(k)$ and the gain perturbation $N(k)$, if the LMIs (8) $\sim$ (10) have feasible solutions such as symmetric positive definite matrices $X \in \Re^{n\times n}$ and $Y \in \Re^{n\times n}$, and positive scalar $\epsilon_i > 0,\ i = 1, 2$, then K is the guaranteed cost control gain matrix. Furthermore, the corresponding value of the cost function (3) satisfies the following inequality (11) for the admissible uncertainties $F(k)$ and the gain perturbations $N(k)$;*

$$J < J^* = x^T(0)X^{-1}x(0) = x^T(0)Yx(0). \tag{11}$$

Proof : *Let us introduce the matrices $X = P^{-1}$ and $Y = P$.*

Using the result in [8], the LMI (8), (9) yields (12). Applying Schur complement [9], and using a standard matrix inequality [10] for the admissible uncertainties and the gain perturbations (2) to the LMI (12), moreover, applying Schur complement to the matrix inequality, it is easy to verity that the LMI (12) satisfies the matrix inequality (4).

On the other hand, since the results of the cost bound (11) can be proved by using the similar argument for the proof of Lemma 1, it is omitted. ■

In this paper, in order to find the matrix pair (X, Y) such that the pair satisfies the LMIs (8) $\sim$ (10) and $X = Y^{-1} > 0$, make use of the following algorithm [11].

Algorithm: *For solving the above problem, the linearization algorithm is conceptually described as follows.*

1) *Find a feasible solution set $(\varepsilon_1^0,\ \varepsilon_2^0,\ X^0,\ Y^0)$ for satisfying the LMIs (8) $\sim$ (10). If there are none, exit. Set r = 0.*

2) *Set $V^r = Y^r, W^r = X^r$ and find X^{r+1}, Y^{r+1} that solve the LMI problem*
Minimize $\mathbf{Tr}(V^rX + W^rY)$ *subject to (8) $\sim$ (10).*

3) *If a stopping criterion is satisfied, exit. Otherwise, set r = r+1 and go to step 2).*

A solution set of $(\varepsilon_1,\ \varepsilon_2,\ X,\ Y)$ is easy to acquire, because the algorithm is simple LMI problem. Moreover, it is shown in [11] that the algorithm converges.

III. NEURAL NETWORKS FOR ADDITIVE GAIN PERTURBATIONS

The LMI approach for the uncertain discrete–time systems usually results in the conservative controller design due to the existence of the uncertainties $F(k)$ and the gain perturbations $N(k)$, which lead the large cost J. The main purpose of this paper is to introduce NN as the additive gain perturbations into the discrete-time uncertain system to improve the cost performance. Note that the proposed neurocontroller regulates its outputs in real–time under the robust stability guaranteed by the LMI approach.

A. On–line learning Algorithm of neurocontroller

It can be much expected that the reduction of the cost will be attain when the uncertain discrete–time system performs the nominal closed–loop system.

Let us consider the following nominal system without uncertainties as:

$$\hat{x}(k+1) = A\hat{x}(k) + B\hat{u}(k), \tag{13a}$$

$$\hat{y}(k) = C\hat{x}(k), \tag{13b}$$

$$\hat{u}(k) = \hat{K}\hat{y}(k), \tag{13c}$$

where $\hat{x}(k) \in \Re^n$ is the state, $\hat{y}(k) \in \Re^m$ is the output and $\hat{u}(k) \in \Re^l$ is the control input. $\hat{K}$ is the output feedback gain derived by the LMI approach [8] for the nominal system (13). For the nominal system (13) and cost function (3), it is known

$$
\begin{bmatrix} 0 \\ B \\ 0 \\ I_r \\ 0 \\ 0 \end{bmatrix}^{\perp}
\begin{bmatrix}
-Y & A^T & I_q & 0 & E_1^T & [E_2C]^T \\
A & -X+\epsilon_1 D_1 D_1^T+\epsilon_2 B D_2 D_2^T B^T & 0 & \epsilon_2 B D_2 D_2^T & 0 & 0 \\
I_q & 0 & -Q^{-1} & 0 & 0 & 0 \\
0 & \epsilon_2 D_2 D_2^T B^T & 0 & -R^{-1}+\epsilon_2 D_2 D_2^T & 0 & 0 \\
E_1 & 0 & 0 & 0 & -\epsilon_1 I_{\epsilon 1} & 0 \\
E_2 C & 0 & 0 & 0 & 0 & -\epsilon_2 I_{\epsilon 2}
\end{bmatrix}
\begin{bmatrix} 0 \\ B \\ 0 \\ I_r \\ 0 \\ 0 \end{bmatrix}^{\perp T} < 0, \tag{8}
$$

$$
\begin{bmatrix} C^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\perp}
\begin{bmatrix}
-Y & A^T & I_q & 0 & E_1^T & [E_2C]^T \\
A & -X+\epsilon_1 D_1 D_1^T+\epsilon_2 B D_2 D_2^T B^T & 0 & \epsilon_2 B D_2 D_2^T & 0 & 0 \\
I_q & 0 & -Q^{-1} & 0 & 0 & 0 \\
0 & \epsilon_2 D_2 D_2^T B^T & 0 & -R^{-1}+\epsilon_2 D_2 D_2^T & 0 & 0 \\
E_1 & 0 & 0 & 0 & -\epsilon_1 I_{\epsilon 1} & 0 \\
E_2 C & 0 & 0 & 0 & 0 & -\epsilon_2 I_{\epsilon 2}
\end{bmatrix}
\begin{bmatrix} C^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\perp T} < 0, \tag{9}
$$

$$
\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} > 0. \tag{10}
$$

$$
\begin{bmatrix}
-P & [A+BKC]^T & I_q & [KC]^T & E_1^T & [E_2C]^T \\
A+BKC & -P^{-1}+\epsilon_1 D_1 D_1^T+\epsilon_2 B D_2 D_2^T B^T & 0 & \epsilon_2 B D_2 D_2^T & 0 & 0 \\
I_q & 0 & -Q^{-1} & 0 & 0 & 0 \\
KC & \epsilon_2 D_2 D_2^T B^T & 0 & -R^{-1}+\epsilon_2 D_2 D_2^T & 0 & 0 \\
E_1 & 0 & 0 & 0 & -\epsilon_1 I_{\epsilon 1} & 0 \\
E_2 C & 0 & 0 & 0 & 0 & -\epsilon_2 I_{\epsilon 2}
\end{bmatrix} < 0. \tag{12}
$$

that the guaranteed cost of the nominal system $\hat{J}^*$ is smaller than that of uncertain system J^* [8].

The NN in the proposed system should be trained in real–time so that the norm of the output discrepancy between the behavior of the nominal system and the uncertain discrete–time system $\left\|\hat{y}(k+1) - y(k+1)\right\|$ becomes as small as possible at each step k. An energy function $E(k)$ is defined as the discrepancy. At each step, the weight coefficients are modified so as to minimize $E(k)$ given as

$$
E(k) \triangleq (\hat{y}(k+1) - y(k+1))^T(\hat{y}(k+1) - y(k+1)). \tag{14}
$$

If $E(k)$ can be minimized as small as possible, the norm of the discrepancy $\left\|\hat{y}(k+1) - y(k+1)\right\|^2$ would also be minimized so that the cost of the uncertain discrete–time system is close to the cost of the nominal system.

In the learning of NN, the modification of weight coefficient, $\Delta w_{ij}(k)$, is given as

$$
\begin{aligned}
w_g^{ij}(k+1) &= w_g^{ij}(k) + \Delta w_g^{ij}(k), && \text{(15a)} \\
\Delta w_g^{ij}(k) &= -\eta \frac{\partial E(k)}{\partial w_g^{ij}(k)}, && \text{(15b)} \\
\frac{\partial E(k)}{\partial w_g^{ij}(k)} &= \frac{\partial E(k)}{\partial y(k+1)} \frac{\partial y(k+1)}{\partial x(k+1)} \frac{\partial x(k+1)}{\partial u(k)} && \text{(15c)} \\
&\quad \times \frac{\partial u(k)}{\partial N(k)} \frac{\partial N(k)}{\partial w_g^{ij}(k)},
\end{aligned}
$$

where η is the learning ratio. The term $\frac{\partial E(k)}{\partial y(k+1)}$, $\frac{\partial y(k+1)}{\partial x(k+1)}$, $\frac{\partial x(k+1)}{\partial u(k)}$, $\frac{\partial u(k)}{\partial N(k)}$ can be calculated from the uncertain system (1) and the energy function (14) as follows:

$$
\begin{aligned}
\frac{\partial E(k)}{\partial y(k+1)} &= -\Big(\hat{y}(k+1) - y(k+1)\Big), \\
\frac{\partial y(k+1)}{\partial x(k+1)} &= C,\ \frac{\partial x(k+1)}{\partial u(k)} = B, \\
\frac{\partial u(k)}{\partial N(k)} &= D_2 E_2 C y(k),
\end{aligned} \tag{16}
$$

and $\frac{\partial N(k)}{\partial w_g^{ij}(k)}$ can be calculated using the chain rule on the NN. From (14) $\sim$ (16), NN can be trained so as to decrease the cost J on–line.

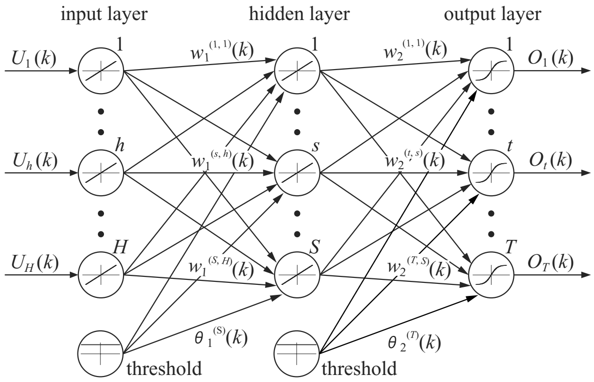

Fig. 2. Structure of the multilayered neural networks.

B. Multilayered Neural networks

The utilized NN are of a three–layer feed–forward network as shown in Fig. 2. A linear function is utilized in the neurons of the input and the hidden layers, and a sigmoid function in the output layer. Inputs and outputs of each layer can be described as follows

$$
s_g^i(k) = \begin{cases}
U_i(k) & \{g = 1(\text{input layer})\}, \\
\sum w_1^{(i,j)}(k) o_1^j(k) & \{g = 2(\text{hidden layer})\}, \\
\sum w_2^{(i,j)}(k) o_2^j(k) & \{g = 3(\text{output layer})\},
\end{cases}
$$

$$
o_g^i(k) = \begin{cases}
s_1^i(k) & \{g = 1(\text{input layer})\}, \\
s_2^i(k) + \theta_1^{(i)}(k) & \{g = 2(\text{hidden layer})\}, \\
\dfrac{1 - e^{(-s_3^i(k) + \theta_2^{(i)}(k))}}{1 + e^{(-s_3^i(k) + \theta_2^{(i)}(k))}} & \{g = 3(\text{output layer})\},
\end{cases}
$$

where $s_g^i(k)$ and $o_g^i(k)$ are the input and output of neuron i in the g–th layer at step k, $w_g^{i,j}(k)$ indicates the weight coefficient from neuron j in the g–th layer to neuron i in the $(g+1)$–th layer, $U_i(k)$ is the input of NN, $\theta_g^{(i)}(k)$ is a positive constant for the threshold of neuron i in the $(g+1)$–th layer. As the additive gain perturbations defined in the formula (2), the outputs of NN are set in the range of $[-1.0,\ 1.0]$.

IV. Numerical Example

In this section, the effectiveness of the proposed method is verified on the discrete–time uncertain system given by

TABLE I
A comparison of the cost in each condition.

The cost of the nominal system $\hat{J}$ = 91.0684

$F(k)$	Learning ratio η	With NN	Without NN
1	0.5	119.2710	138.0183
$\exp(-0.5k)$	0.2	116.5204	122.7675
$\cos(0.5\pi k)$	0.05	115.0893	116.6713

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.2 & 0 \end{bmatrix}, \quad D_2 = 0.15,$$
$$E_2 = 1, \quad F(k) = \exp(-0.5k), \quad N(k) = N_1(k)$$

where $N_1(k)$ is the outputs of NN. The initial system condition is $x(0) = [5\ 5]^T$, and the weighting matrices are chosen as $Q = \mathrm{diag}(1, 2)$ and $R = 1$, respectively.

The output feedback control gain K based on the proposed LMI design method with a neurocontroller is given by

$$K = -0.5036. \tag{17}$$

For the nominal system (13), the output feedback control gain $\hat{K}$ based on the LMI design method in [8] is given by:

$$\hat{K} = -0.4794. \tag{18}$$

For the system without the proposed neurocontroller, that is $N(k) \equiv 0$, the control input of the uncertain system is described as

$$u(k) = \bar{K}y(k), \tag{19}$$

where the output feedback control gain $\bar{K}$ is designed based on the LMI approach [8, 12] as

$$\bar{K} = -0.4644. \tag{20}$$

The neurocontroller is composed of 30 neurons in the hidden layer, and one neurons in the input and the output layers, respectively. The output variables are used as the NN' inputs and the learning ratio $\eta = 0.2$. The initial weights are randomly set in the range of $[-0.05, 0.05]$.

The cost J with the gain matrix K is 116.5204, while the cost without the neurocontroller $\bar{J}$ with $\bar{K}$ is 122.7675. Then, The cost of the nominal system $\hat{J}$ is 91.0684. Various uncertain systems were examined by changing $F(k)$. Table I shows that the cost of the proposed system is smaller than that of the system without the neurocontroller in all cases.

The simulation results ($F(k) = \exp(-0.5k)$) are shown in Fig. 3. The response of the proposed neurocontroller is stabilized faster than that without the neurocontroller (Fig. 3 (a)∼(c)). Fig. 3(d) shows the output feedback gain with the additive gain $K + \tilde{K}$, i.e., $K + D_2N(k)E_2$. The response of the proposed one is also stabilized faster than that of the controller without one. The proposed neurocontroller could reduce the cost and compensate for the uncertainties of the system.

Thus, $K + \tilde{K}$ changes so that the response of uncertain system can be close to the one of nominal system, and reduction of the cost performance can be attained for uncertain system. Therefore, the energy function $E(k)$ is adequate for the learning algorithm.

V. Conclusions

The application of neural networks to the output guaranteed cost control problem of the discrete–time uncertain system has been investigated. Using the LMI technique, the class of the output feedback gain for the uncertain system has been derived. Substituting the neurocontroller into the gain perturbations, the robust stability of the closed–loop system is guaranteed even if the systems include NN. Furthermore, by combining linear controllers and NN, the reduction of the cost is attained. Simulation results have shown that the NN have succeeded in reducing the large cost caused by the LMI.

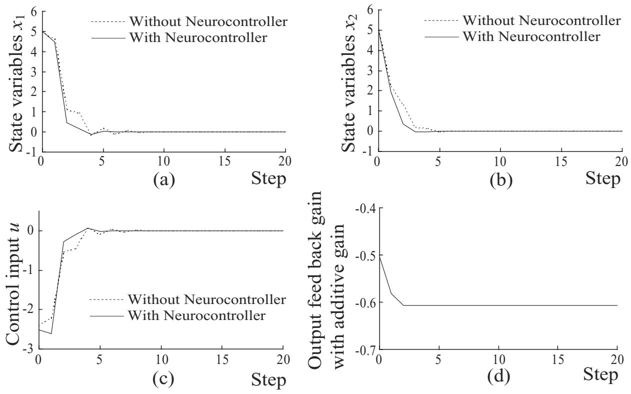

Fig. 3. Simulation results by using the neurocontroller ($F = \exp(-0.5k)$). (a), (b) State variables. (c) Control input. (d) Output feedback gain with additive gain.

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