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Multi-Arm/Finger Grasping: One View to the Stability Problem

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Abstract

The paper deals with the rotational stability of a rigid body under the constant internal forces. For this problem, first, the stiffness tensor is constructed and its basic properties are analyzed. The internal force parameterization is done with the use of the virtual linkage/spring model. Within this parameterization, necessary and sufficient conditions of stability are obtained in the analytical form. In the space of the internal forces they form a region given by intersection of a plane and a singular quadric. Since the stability conditions guarantee only positive definiteness of the stiffness tensor, the contact friction is taken into account separately. In this paper analysis of the unilateral constraints is done under a study case, where achieving stable grasp of a convex object, with the stretching internal forces created by friction, is studied in an analytical example.

1 Introduction

One of the fundamental problems in controlling multi-fingered hands is stability of the resulting grasp. In recent years, the problem has been addressed from different points of view and a number of approaches to defining the grasping stability, its robustness, and relation to such concepts as grasping form and force closure, has been proposed in literature. A very good survey on this topic can be found in [1]. Here in this paper we address the problem in a somewhat simplified way, dealing only with the rotational stability of the object.

Basically, the total compliance of the multi-finger system has two sources. The first one is due to compliance of the finger itself, and the second one is due to the contact force interaction between the finger and the object. The Cartesian compliance of the fingers is symmetric and positive definite (and therefore stable) as long as the joint compliance matrix is stable. The compliance due to the interaction depends on the contact force distribution, and is often the source of grasping instability. This phenomena has been discovered and studied in [2–4]. It should be noted that a very similar subject—stability due to internal forces in mechanisms with closed kinematic chains—was analyzed in [5, 6].

One possible approach to provide stable grasping is to design the total compliance matrix to be positive definite. Theoretically this approach can work nicely. However, it is case-dependent approach and there is no systematic procedure for adjusting the compliance of the fingers to that of the object. Another possible solution of the stability problem is based on decomposition of the total compliance and designing the corresponding matrices separately. Indeed, if they are both stable than the resulting compliance will also be stable. Conceptually, this approach is taken in this paper, and in our work we do not consider the compliance of the fingers at all. In so doing, we consider the grasp to be stable if its stiffness matrix is not negative definite. The reason for taking this view is simple—even if the contact force induced compliance is positive semidefinite, the resulting compliance of the system can be easily made stable with simple proportional control of the fingers' joints.

Our work was motivated mainly by papers [7], where fundamental properties of the grasp stiffness matrix were under investigation, and by [8], where the rotational stability of the grasped object was analyzed from the classical standpoints of the Lyapunov's theory. However, the relation of the stability to the internal force distribution has not been studied in detail, and necessary and sufficient conditions of stability have not been obtained in the analytical form. In our work we want to fill the gap between the two cited papers. Another papers that are worth noting in this introduction are [9], where the geometric properties of the object has been brought into the stability analysis, and [10], where gravitational effects in the stability problem has been investigated.

This paper is organized as follows. Firstly, in Section 2 the analytical expression for the stiffness tensor is derived and its basic properties are discussed. Force parameterization for the so-called virtual linkage/spring model is given in Section 3. Necessary and sufficient conditions of positive definiteness of the stiffness tensor are derived in Section 4. Combination of the stability conditions with unilateral constraints on the normal reaction forces is studied on an analytical example in Section 5. Finally, conclusions are presented in Section 6.

2 Stiffness Tensor

Let us consider a rigid body subjected to multiple frictional contacts. Assume that the constant forces $\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n$ are applied at the points defined by the radius-vectors $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_n$ drawn from the center of mass O. Gravity is ignored since only the internal forces are studied in this paper. The body is at the equilibrium so the static equations read $\sum_{i=1}^{n} \boldsymbol{f}_i = \boldsymbol{0}$, $\sum_{i=1}^{n} \boldsymbol{\rho}_i \times \boldsymbol{f}_i = \boldsymbol{0}$. Let $\boldsymbol{\theta} = 2\boldsymbol{e} \tan \frac{\chi}{2}$ be the vector of the finite rotation of the body [11]. Here, $\boldsymbol{e}$ defines the axis of rotation, and χ stands for the rotation angle. The coordinates of the contact points after rotation of the body, $\boldsymbol{\rho}_i(\boldsymbol{\theta})$, are defined by the Rodriguez' equation for the finite rotation:

$$\boldsymbol{\rho}_i(\boldsymbol{\theta}) = \boldsymbol{\rho}_i + \frac{1}{1 + \frac{1}{4}\boldsymbol{\theta}^T\boldsymbol{\theta}}\, \boldsymbol{\theta} \times (\boldsymbol{\rho}_i + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\rho}_i). \quad (1)$$

Since the contact forces are assumed to be constant, calculating the potential energy leads to the following expression:

$$\Pi = -\sum_{i=1}^{n} \boldsymbol{f}_i^{\mathrm{T}}\{\boldsymbol{\rho}_i(\boldsymbol{\theta}) - \boldsymbol{\rho}_i\} = \frac{1}{2}\, \frac{1}{1 + \frac{1}{4}\boldsymbol{\theta}^T\boldsymbol{\theta}}\, \boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\theta}, \quad (2)$$

where

$$\boldsymbol{K} = \sum_{i=1}^{n} (\boldsymbol{\rho}_i^{\mathrm{T}} \boldsymbol{f}_i)\boldsymbol{I} - \boldsymbol{\rho}_i \boldsymbol{f}_i^{\mathrm{T}} \quad (3)$$

is the stiffness tensor. The potential energy is always positive (and the equilibrium is stable) as long as $\boldsymbol{K}$ is positive definite. Note that for the small rotation $\boldsymbol{\theta}$ the potential energy is transformed to the familiar quadratic form. Also note that in the planar case the rotational stiffness is a scalar given by $K = \sum_{i=1}^{n} \boldsymbol{\rho}_i^{\mathrm{T}} \boldsymbol{f}_i$.

The following properties of the stiffness tensor can be formulated straight away. First, $\boldsymbol{K}$ is symmetric as long as the object is at equilibrium. Indeed, one can prove that $\boldsymbol{K} - \boldsymbol{K}^{\mathrm{T}} = \boldsymbol{\Omega}(\sum_{i=1}^{n} \boldsymbol{\rho}_i \times \boldsymbol{f}_i)$, and, therefore, the skew-symmetric part of $\boldsymbol{K}$ is always zero by the static equations. Here, $\boldsymbol{\Omega}$ is the skew-symmetric operator such that $\boldsymbol{\Omega}(\boldsymbol{a}) \cdot \boldsymbol{b} \equiv \boldsymbol{a} \times \boldsymbol{b}$. If, however, the object is not at equilibrium, $\boldsymbol{K}$ is always asymmetric.

Next, even though $\boldsymbol{K}$ is symmetric at the equilibrium, it is not always and not necessarily positive definite. It is only in some simple cases when the judgment on the positive definiteness of $\boldsymbol{K}$ can be done from the structure of the stiffness tensor (3). Consider, for example, the case when all the applied forces are coplanar to the correspondent vectors $\boldsymbol{\rho}_i$. In this case $\boldsymbol{f}_i = k_i \boldsymbol{\rho}_i$, and formula (3) gives $\boldsymbol{K} = \sum_{i=1}^{n} k_i\{(\boldsymbol{\rho}_i^{\mathrm{T}} \boldsymbol{\rho}_i)\boldsymbol{I} - \boldsymbol{\rho}_i \boldsymbol{\rho}_i^{\mathrm{T}}\}$. As can be seen, $\boldsymbol{K}$ has the structure of the inertia tensor of a system of points built on the vectors $\boldsymbol{\rho}_i$, with k_i playing the role of masses. Therefore, if all $k_i \geq 0$, i.e., all the forces are stretching, $\boldsymbol{K}$ is positive definite and the equilibrium is stable. In the opposite case when all $k_i \leq 0$, i.e., all the forces are compressive, $\boldsymbol{K}$ is negative definite and the equilibrium is unstable.

However, in the general case when k_i have different signs or if the applied forces $\boldsymbol{f}_i$ are not coplanar to $\boldsymbol{\rho}_i$, it is not that easy to to make a judgment on the properties of $\boldsymbol{K}$ without its direct computing, and additional study of the force structure is required.

Finally, please note that the forces we deal with in this paper are assumed to be constant in the inertial frame. If they are constant in the body frame, we can show that in such a case they do not contribute to the stiffness tensor as long as the body is in the equilibrium.

3 Force Parameterization

To relate the stability properties to the structure of the applied forces one should make the force decomposition and obtain an analytical solution of the static equations. The valid solution can be based on the pseudo-inversion of the grasp matrix. Introducing the block vector $\boldsymbol{f} = \{\boldsymbol{f}_1^{\mathrm{T}}, \ldots, \boldsymbol{f}_n^{\mathrm{T}}\}^{\mathrm{T}}$, we can rewrite the static equations in the following form: $\boldsymbol{B}_o \boldsymbol{f} = \boldsymbol{0}$, where

$$\boldsymbol{B}_o = \left\| \begin{matrix} \boldsymbol{I} & \cdots & \boldsymbol{I} \\ \boldsymbol{\Omega}(\boldsymbol{\rho}_1) & \cdots & \boldsymbol{\Omega}(\boldsymbol{\rho}_n) \end{matrix} \right\|. \quad (4)$$

The general solution of the statics equations can be represented as $\boldsymbol{f} = \boldsymbol{P}_I \boldsymbol{\varphi}$, where $\boldsymbol{P}_I = \boldsymbol{I} - \boldsymbol{B}_c^{+}\boldsymbol{B}_c$ is the orthogonal projector onto the null space of the grasp matrix $\boldsymbol{B}_o$, i.e., onto the space of the internal forces, and it does not depend on the reference point $\boldsymbol{\rho}_c$. Here, $\boldsymbol{\varphi} = \{\boldsymbol{\varphi}_1^{\mathrm{T}}, \ldots, \boldsymbol{\varphi}_n^{\mathrm{T}}\}^{\mathrm{T}}$ is composed of the arbitrary specified vectors $\boldsymbol{\varphi}_i$.

Note that $\boldsymbol{\varphi}$ defines redundant representation of the internal forces and does not have clear physical meaning. To introduce physically meaningful parameterization of the internal forces, let us, following to [12], characterize the interaction between any two fingers by

$$\alpha_{ij} = (\boldsymbol{r}_i - \boldsymbol{r}_j)^{\mathrm{T}}(\boldsymbol{f}_i - \boldsymbol{f}_j), \quad (5)$$

i.e., by the difference of the contact forces projected along the line joining the two contact points. The interaction force is of compression type if $\alpha_{ij} < 0$, and of tension type if $\alpha_{ij} > 0$. The physical meaning of α_{ij} is the work produced by $\boldsymbol{f}_{ij} = \boldsymbol{f}_i - \boldsymbol{f}_j$ on the displacement $\boldsymbol{r}_{ij} = \boldsymbol{\rho}_i - \boldsymbol{\rho}_j$.

Note that the dimension of α_{ij}, [N · m], can also be interpreted as that of the rotational stiffness. Continuing this thought, we could have introduced another possible parameterization of the internal forces by $\tilde{\alpha}_{ij} = \alpha_{ij}/(\boldsymbol{r}_{ij}^{\mathrm{T}}\boldsymbol{r}_{ij})$, where $\tilde{\alpha}_{ij}$ are the stiffness of the linear virtual spring connecting the two contact points. It is remarkable that in this interpretation the grasp of the rigid body can be represented by the virtual springs which can have as positive as well as negative stiffness. It should also be noted that this interpretation is closed conceptually to the virtual linkage model considered in [13]. In the forthcoming analysis, for the sake of simplicity of the resulting mathematical expressions, we, however, will deal with the parameterization given by (5).

In the non-redundant, minimal representation of the internal forces, for which there exists one-to-one mapping between the applied forces $\boldsymbol{f}$ and the vector combined of $\boldsymbol{\Phi}_o$ and $\boldsymbol{\alpha}$, the solution of the static equations is specified as

$$\boldsymbol{f} = \boldsymbol{P}_{I\alpha}\boldsymbol{\alpha}, \quad (6)$$

where $P_{I\alpha} \in \Re^{2n\times(2n-3)}$ in the planar case, and $P_{I\alpha} \in \Re^{3n\times3(n-2)}$ in the spatial case. Note that $\alpha = \{\alpha_{ij}\} \in \Re^{\frac{n(n-1)}{2}}$. Equating the dimension of α to the column-dimension of $P_{I\alpha}$, one obtains the numbers of the contact points admissible for the minimal representation. They are $n = 2$ or $n = 3$ in the planar case, and $n = 3$ or $n = 4$ in the spatial case.

To obtain an analytical expression for the matrix $P_{I\alpha}$, one must represent (5) in the matrix form so that $\alpha = A_o(r_{ij})f$. This representation depends on how the elements of α are ordered. Upon constructing A_o, we can prove that the matrix $P_{I\alpha}$ can be expressed in the following form:

$$P_{I\alpha} = A_o^+ = A_o^T(A_oA_o^T)^{-1}. \quad (7)$$

This formula and the representation (6) also remain true for $n \leq 7$, i.e., even though parameterization of the internal forces in terms of α becomes redundant. It, however, still keeps the advantage of having clear physical meaning, and is by no means worse than the parameterization by φ.

4 Analysis of Stability

Let us now consider stability due to the internal forces. To facilitate mathematical description of the forthcoming analysis and to cover the general case of n contact points, we will use another description for the elements α_{ij} of the vector α. Namely, it will be assumed that they are somehow ordered and can be addressed by only one subscript. The same rule will be kept for the correspondent vectors r_{ij} and f_{ij}. The use of single indexed variables will be marked by bar sign, i.e., $\bar{\alpha}_l$ will correspond to some element α_{ij}.

We start the analysis with considering the planar case. First, by direct summing up all α_{ij} as given by (5), we arrive at the following remarkable formula

$$K = \frac{1}{n}\sum_{i=1}^{N}\bar{\alpha}_i, \quad (8)$$

where $N = n(n-1)/2$ is dimension of the vector α. Hence, for the planar case the condition of $\sum_{i=1}^{N}\bar{\alpha}_i \geq 0$ is necessary and sufficient for stability under the internal forces.

Next, coming to the spatial case, we represent the internal forces as $f = A_o^+\alpha$ and substitute them into (3). After simplification we obtain

$$K = \sum_{i=1}^{N}\bar{\beta}_i\{(\bar{r}_i^T\bar{r}_i)I - \bar{r}_i\bar{r}_i^T\}, \quad (9)$$

where elements of the vector $\beta = \{\bar{\beta}_1, \ldots, \bar{\beta}_N\}^T$ are related to the components of the vector α by $\beta = (A_oA_o^T)^{-1}\alpha$. As can be easily seen, K has the structure of the inertia tensor of a system of points built on the vectors $\bar{r}_i$, with $\bar{\beta}_i$ playing the role of masses. Hence, N linear with respect to $\bar{\alpha}_i$ conditions of $\bar{\beta}_i \geq 0$ would be sufficient to guarantee that the matrix K is not negative definite. As will be shown below, they can be reduced to just two conditions—one linear and the other one quadratic—imposed on the elements of α.

At first, we will show that $\lambda_1 = \frac{1}{n}\sum_{i=1}^{N}\bar{\alpha}_i$ is valid eigenvalue of the matrix K. Indeed, computing the determinant of the matrix $K - \lambda_1 I$, we get

$$\det(K - \lambda_1 I) = \det(\sum_{i=1}^{N}\bar{\beta}_i\bar{r}_i\bar{r}_i^T). \quad (10)$$

By the generalized Lagrange identity [14] we have

$$\det(\sum_{i=1}^{N}\bar{\beta}_i\bar{r}_i\bar{r}_i^T) = \frac{1}{6}\sum_{i,j,k=1}^{N}\bar{\beta}_i\bar{\beta}_j\bar{\beta}_k(\det[\bar{r}_i\vdots\bar{r}_j\vdots\bar{r}_k])^2. \quad (11)$$

However, by construction of the virtual spring model, every three vectors $\bar{r}_i, \bar{r}_j, \bar{r}_k$ there are linearly dependent. Hence, $\det(K - \lambda_1 I) = 0$ and λ_1 is the eigenvalue of K_I. Thus, the condition $\sum_{i=1}^{N}\bar{\alpha}_i \geq 0$ is also necessary for stability in the spatial case.

As to the other two eigenvalues of K, we note that $\mathrm{tr}K = 2\lambda_1$, and therefore

$$\lambda_{2,3} = \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - 4\gamma(\alpha)}}{2}, \quad (12)$$

where $\gamma(\alpha) = \det K/\lambda_1$ is a quadratic form of α. The stability is guaranteed if $\gamma = \alpha^T\Gamma_\alpha\alpha \geq 0$. Now, the remaining part of the analysis is to define the matrix Γ_α. Note here that γ can be also represented in terms of variables β, i.e., as $\gamma = \beta^T\Gamma_\beta\beta$. We can define elements of Γ_β with the use of the unit coefficients method, i.e., by computing $\det K$ and λ_1 with the vectors $\beta = \{0, \ldots, 0, 1, 0, \ldots, 0\}^T$ and $\beta = \{0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0\}^T$. Skipping all the intermediate calculations, we can show that

$$\{\Gamma_\beta\}_{ij} = \begin{cases} 0 & \text{if } i = j \\ (\bar{r}_i \times \bar{r}_j)^T(\bar{r}_i \times \bar{r}_j) & \text{otherwise} \end{cases} \quad (13)$$

It is interesting that the geometric meaning of the off-diagonal elements of Γ_β is that $\{\Gamma_\beta\}_{ij} = S_{ij}^2/4$, where S_{ij} is the area of the triangle built on the vectors $\bar{r}_i$ and $\bar{r}_j$. Note that Γ_β is singular sign-indefinite form. We can show that in the space of variable β this quadric is represented by a cone.

Now, having defined Γ_β we can return to Γ_α. Taking into account the relation between α and β, we obtain

$$\Gamma_\alpha = (A_oA_o^T)^{-1}\Gamma_\beta(A_oA_o^T)^{-1} \quad (14)$$

Finally, recalling the relation between α and f, we can formulate the second stability condition in terms of the internal forces f. It reads

$$f^T A_o^+\Gamma_\beta(A_o^+)^T f \geq 0. \quad (15)$$

This completes our analysis.

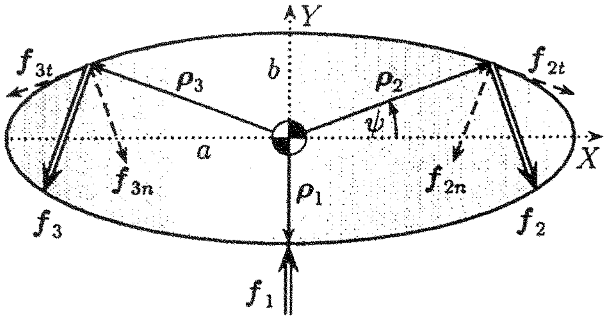

Figure 1: Example of three-fingered grasp.

5 Stability Under Unilateral Constraints: A Study Case

Let us consider an example of grasping of ellipse by a three-fingered hand. To simplify the forthcoming analysis, we assume symmetrical placement of the second and third contact points on the object as shown in Fig. 1. Under this assumption, the contact points are defined as follows: $\rho_1 = \{0, -b\}^T, \rho_2 = \{a\cos\psi, b\sin\psi\}^T, \rho_3 = \{-a\cos\psi, b\sin\psi\}^T$, where the grasping angle $-\pi/2 \leq \psi \leq \pi/2$, and a and b are the lengths of the semiaxes of the ellipse. The above-made simplification will allow us to obtain analytical expressions for the rotational stiffness and conduct the stability analysis in analytical form. The normal contact forces, directed along the inward normals, are defined as follows: $f_{1n} = f_y\{0,1\}^T, f_{2n} = f_n^*\{-b\cos\psi, -a\sin\psi\}^T, f_{3n} = f_n^*\{b\cos\psi, -a\sin\psi\}^T$, where $f_n^* = f_n/\lambda(\psi)$, and $\lambda(\psi) = \sqrt{a^2\sin^2\psi + b^2\cos^2\psi}$. Defining clockwise (for the 2nd force) and counter clockwise (for the 3rd force) tangential vectors, we specify the friction forces in the following form: $f_{1t} = \{0,0\}^T, f_{2t} = f_t^*\{a\sin\psi, -b\cos\psi\}^T, f_{3t} = f_t^*\{-a\sin\psi, -b\cos\psi\}^T$, where the normalized tangential force $f_t^* = f_t/\lambda(\psi)$.

Under the above-specified contact forces the moment balance and the force balance for the horizontal axis are always satisfied. The static equation for the force balance for the vertical axis reads

$$f_y = 2f_n^* a\sin\psi + 2f_t^* b\cos\psi. \tag{16}$$

Next, we should introduce the unilateral constraints on the normal forces $f_y \geq 0$ and $f_n \geq 0$. Under these constraints the equilibrium region in the plane $f_t/f_n, \psi$ is defined as

$$f_t/f_n \geq \mu_y(\psi) = -a\sin\psi/b\cos\psi, \tag{17}$$

with $f_t/f_n = \mu_y(\psi)$ defining the line of zero internal force f_y. Finally, the Coulomb friction constraints

$$-\mu_c \leq f_t/f_n \leq \mu_c, \tag{18}$$

are taken into consideration. Here, μ_c stands for the friction coefficient.

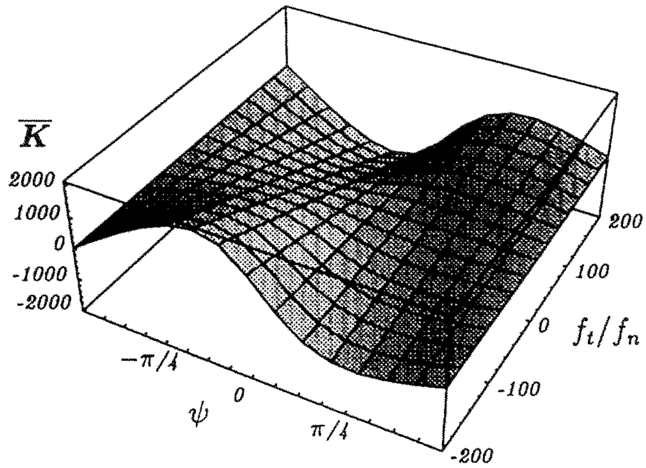

Figure 2: Normalized rotational stiffness.

Having specified the contact points and the contact forces, we calculate the rotational stiffness

$$K = -bf_y - 2abf_n^* + 2(a^2 - b^2)\sin\psi\cos\psi f_t^*. \tag{19}$$

Since $f_y \geq 0$ and $f_n \geq 0$, the rotational stiffness is not positive if the object is sphere ($a = b$) or if there is no friction forces.

To proceed further with the stability analysis, we substitute (16) into (19) and represent

$$K = -2ab(1+\sin\psi)f_n^* + 2\cos\psi\{\sin\psi(a^2-b^2)-b^2\}f_t^* \tag{20}$$

as a function of three variables f_n^*, f_t^*, and ψ. We, however, can inspect basic features of this function by plotting, for some fixed values of a and b, the normalized (with respect to the normal force) stiffness $\overline{K} = K/f_n$ as a function of the grasping angle ψ and the normalized friction force f_t/f_n. As can be seen from Fig. 2, it is a sign-indefinite function having a saddle point in the origin.

Let us define the parameters of the grasp under which the equilibrium is stable. To facilitate the analysis we introduce the following function:

$$\mu_s(\psi) = \frac{ab(1+\sin\psi)}{\cos\psi\{\sin\psi(a^2-b^2)-b^2\}}. \tag{21}$$

Note that $f_t/f_n = \mu_s(\psi)$ defines the so-called zero stiffness line in the plane $f_t/f_n, \psi$, which is the zero level curve of the surface $\overline{K}(f_t/f_n, \psi)$.

Let us firstly examine the case of $\psi < 0$. It follows from (17) that $f_t \geq 0$ must be held for keeping the equilibrium. However, in this case $(a^2 - b^2)\sin\psi < b^2$ for any values of a and b, and formula (20) give us negative stiffness. Next, consider the case of $a < b$ and $\psi > 0$. Here we have $(a^2 - b^2)\sin\psi < b^2$. It follows from

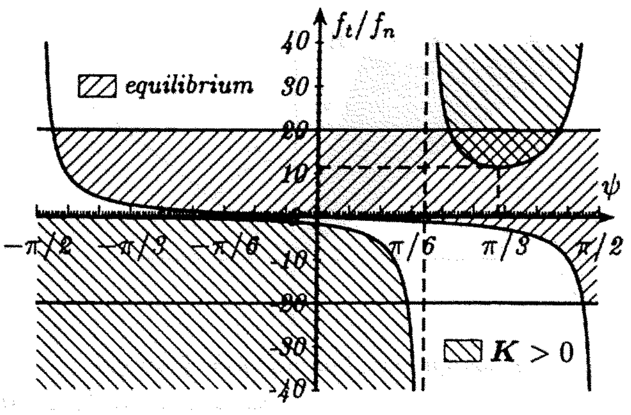

Figure 3: Stable equilibrium: $a > \sqrt{2}\,b$.

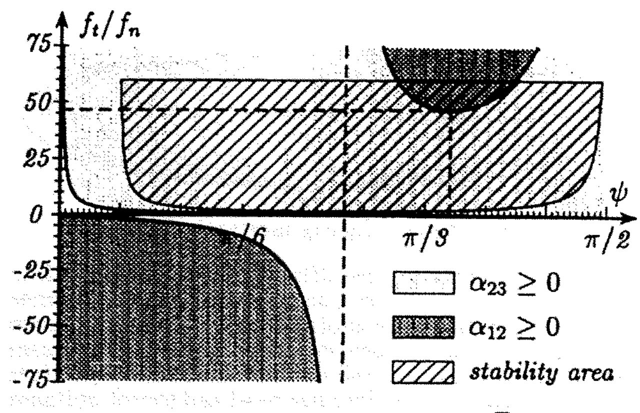

Figure 4: Internal forces: $a > \sqrt{6}\,b$.

(19) that $f_t \leq 0$ must be held. Combining (17) with the condition of $K > 0$ we get $\mu_y \leq f_t/f_n \leq \mu_s$. As can be easily shown, this double inequality does not have solutions unless $(a^2 - b^2)\sin\psi > b^2$, which is impossible under the given choice of parameters. For the case of $b < a < \sqrt{2}\,b$ and $\psi > 0$ we have $(a^2 - b^2)\sin\psi < b^2$. It follows from (19) that $f_t \geq 0$ must be held for attaining stability. This, however, leads to negative stiffness in (20). In all these cases the stability area and the area of possible equilibriums have no intersection.

Finally, we consider the case of $a > \sqrt{2}\,b$ and $\psi > 0$. Here $f_t \geq 0$ is necessary for K to be positive, and this eliminates from consideration the grasping angles for which $(a^2 - b^2)\sin\psi < b^2$. Thus, we have shown that stability of the grasp can be attained if the following restrictions on the shape of the object and on the grasping angle are imposed:

$$a > \sqrt{2}\,b \quad \& \quad \psi \geq \arcsin b^2/(a^2 - b^2). \qquad (22)$$

Taking into account the Coulomb friction constraints (18), the stability region in the plane $f_t/f_n, \psi$ under conditions (22) is defined by the following double inequality

$$\mu_s(\psi) \leq f_t/f_n \leq \mu_c, \qquad (23)$$

which can be called as the task stability cone. The stability region is shown in Fig. 3. Note that from equation $\mu(\psi) = \mu_c$ we can estimate the minimum and the maximum grasping angles for a given friction coefficient μ_c and for a given object shape a, b.

As can be seen from Fig. 3, if the friction coefficient is small enough so that $\mu_c < \mu_s(\psi)$ for all ψ in the region (22), the stable equilibrium is unreachable. Thus, minimum of $\mu_s(\psi)$ sets the critical value of the friction coefficient, μ_{crit}. Solving equation $d\mu_s(\psi)/d\psi = 0$, we can show that this minimum is attained under the following critical grasping angle:

$$\sin\psi_{\text{crit}} = \frac{-1 + \sqrt{\frac{5a^2 - b^2}{a^2 - b^2}}}{2}. \qquad (24)$$

It is interesting to see how the shape of the ellipse affects these critical values. To this end we introduce dimensionless parameter $z = b/a$. It can be shown that μ_{crit} is monotonic function of z on the interval $0 \leq z \leq 1/\sqrt{2}$. It goes to infinity as $z \to 1/\sqrt{2}$, i.e., $\psi_{\text{crit}} \to \pi/2$. In the other limiting case of $z \to 0$, when $a \gg b$, $\mu_{\text{crit}} \to 0$, i.e., no friction is necessary. Here we have $\sin\psi_{\text{crit}} = (-1 + \sqrt{5})/2$. It is remarkable that in this case the y coordinate of the 2nd and 3rd contact points divides the semiaxis b of the ellipse in the golden section ratio $(1 + \sqrt{5})/2$, and the x coordinate of those points divides the semiaxis a in square of the golden section ratio. It is shown in Fig. 6.

Having completed the stability analysis, let us now see the behavior of the internal forces. In the case under study we have $\alpha_{12} = \alpha_{31}$ due to the symmetry of the 2nd and 3rd contact points. Hence, $K = \alpha_{23} + 2\alpha_{12}$. Direct calculation by (5) gives $\alpha_{23} = 4a\cos\psi(f_t^* a\sin\psi - f_n^* b\cos\psi)$, $\alpha_{12} = \alpha_{23}/4 - 3b(1 + \sin\psi)f_y/2$. To analyze the internal forces in the region of stability, we introduce the following functions: $\mu_{23}(\psi) = b\cos\psi/a\sin\psi$, $\mu_{12}(\psi) = ab(1 + \sin\psi)(1 + \sin 2\psi)/(\cos\psi\{\sin\psi(a^2 - 3b^2) - 3b^2\})$, defining the lines of zero internal forces in the plane $f_t/f_n, \psi$.

First, it can be shown that if $\psi > 0$ then $\mu_s(\psi) > \mu_{23}(\psi)$, and therefore $\alpha_{23} > 0$ in the stability region. Thus, in stable grasp 2nd and 3rd forces produce stretching internal force, and, intuitively, this is well comprehended. Also intuitively, it does not seem likely to achieve stable grasping with all the internal forces being of stretching type*. However, as it is shown below, this first-glance impression is wrong.

Indeed, for the case of $(a^2 - 3b^2)\sin\psi < 3b^2$ the condition $f_t/f_n < \mu_{12}(\psi)$ is not consistent with the statics constraint $f_t/f_n > \mu_y(\psi)$. Thus, $\alpha_{12} < 0$ and the corresponding internal force is of compressive type.

*This is because of the unilateral character of constraints imposed on the contact forces, which makes an essential difference between multi-finger and, say, parallel mechanism manipulations.

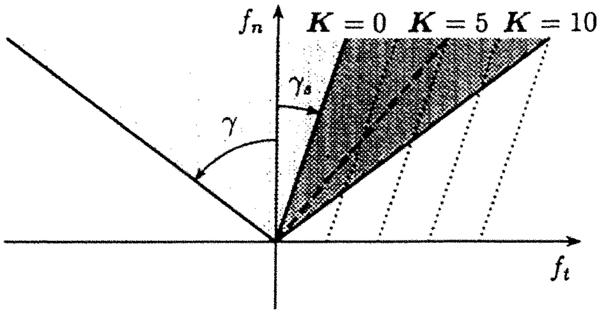

Figure 5: Friction and stability cones.

However, in the case of

$$a > \sqrt{6}\,b \quad \& \quad \psi \geq \arcsin 3b^2/(a^2 - 3b^2) \tag{25}$$

α_{12} becomes positive and all the internal forces are stretching ones. The region of the all-positive internal forces is shown in Fig. 4. Note that accessibility of this region is defined by the friction coefficient higher than the one necessary for just attaining stability.

Finally, we would like to make some comments on the practical force distribution scheme. In the context of the study under consideration it means finding such a normal force that the resulting grasp would be stable. Assuming that ψ is fixed, the acceptable normalized friction force is defined by the stability cone (23). It is shown in Fig. 5, where γ_c and γ_s are the angles corresponding to μ_c and μ_s. Reaching any boundaries of this cone is equally undesirable since they define the sliding line and the critical stability line. In this situation, the most simple solution is to put

$$f_t/f_n = (\mu_s(\psi) + \mu_c)/2, \tag{26}$$

i.e., to set it on the middle line (dashed line in Fig. 5) of the stability cone. Now, with f_t/f_n being fixed, we can choose the normal reaction f_n from prespecified value of the desired stiffness K_{des}. It defines

$$f_n = \frac{K_{\text{des}}\,\lambda(\psi)}{-ab(1+\sin\psi) + \mu_c \cos\psi\{(a^2-b^2)\sin\psi - b^2\}} \tag{27}$$

as a function of the grasping angle, the desired rotational stiffness, and the friction coefficient.

As to the optimal choice of the grasping angle, the value of ψ_{crit} is a good candidate. Indeed, due to possible errors in realization of the force control schemes, it is reasonable to set interval (23) as large as possible. And this corresponds to setting such ψ that gives minimum to $\mu_s(\psi)$, i.e, $\psi = \psi_{\text{crit}}$. Grasping configurations corresponding to ψ_{crit} are shown in Fig. 6.

6 Conclusions

The problem of stability of a grasp under the internal force loading has been addressed in this paper. The potential function of the system has been derived,

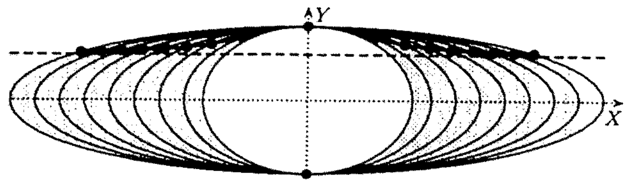

Figure 6: Optimal grasping configurations.

and the structure of the stiffness tensor has been represented through the contact force decomposition. Necessary and sufficient conditions for a stable grasp under internal forces has been derived. Combination of the stability conditions with unilateral constraints on the reaction forces has been studied on an analytical example.

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